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# On an Obstruction to the Hasse Norm Principle and the Equality of Norm Groups of Algebraic Number Fields

Leonid Stern

*Department of Mathematics, Towson University, Baltimore, Maryland 21252*

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Let $L/k$ and $T/k$ be finite extensions of algebraic number fields. In the present work we introduce the factor group of $k^* \cap N_{L/k} J_L N_{T/k} J_T$ by $(k^* \cap N_{T/k} J_T)\, N_{L/k} L^*$, where $J_L$ and $J_T$ are the idele groups of $L$ and $T$, respectively. The main theorem [illegible] at a finite number of primes of the base field $k$. We then apply the main theorem to establish a number of interesting results on the equality of norm groups as subgroups of the multiplicative group of $k$. In particular, we obtain new results on solitary non-Galois extensions of algebraic number fields. 



## INTRODUCTION

Let $L/k$ be a finite extension of algebraic number fields. We will denote by $J_k$ and $C_k$ the idele group and the idele class group of $k$, respectively. Also, we set $N(L/k) = k^* \cap N_{L/k} J_L$, where $N_{L/k}$ is the norm operator. It follows that $N(L/k) = \bigcap_v (k^* \cap [N_{L/k} L^*]_v)$ (see for instance [11, Proposition 1.3, p. 113]), where $v$ ranges over all primes of $k$, and $[N_{L/k} L^*]_v$ is the topological closure of $N_{L/k} L^*$ in $k_v^*$ (the multiplicative group of the completion of $k$ at $v$). Furthermore, the factor group of $N(L/k)$ by $N_{L/k} L^*$ is finite. This factor group is called the *total obstruction to the Hasse Norm Principle* (*HNP*) *for* $L/k$, and its order is denoted by $i(L/k)$. If $i(L/k) = 1$, then we say that HNP holds for $L/k$. Let $E$ be a finite Galois extension of $k$ containing $L$. The factor group of $N(L/k)$ by $N(E/k)\, N_{L/k} L^*$ is a homomorphic image of the total obstruction to HNP for $L/k$. This factor group is called the *first obstruction to HNP for* $L/k$ *corresponding to the tower* $k \subseteq L \subseteq E$ [3]. By Theorem 1 of [3, p. 305] the first obstruction to HNP can be described in terms of Galois groups of local extensions. We generalize the notion of the first obstruction to HNP as follows. Let $T/k$ be an arbitrary finite extension. The factor group of $k^* \cap N_{L/k} J_L N_{T/k} J_T$ by $N(T/k)\, N_{L/k} L^*$ is called the *first obstruction to HNP for* $L/k$ *corresponding to the extension*



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$T/k$. If $T$ is a finite Galois extension of $k$ containing $L$, then the first obstructions to HNP for $L/k$ corresponding to the tower $k \subseteq L \subseteq T$ and to the extension $T/k$ coincide. We will show that the first obstruction to HNP for $L/k$ corresponding to an arbitrary finite extension $T/k$ can be described in terms of Galois groups of local extensions (Theorem 1.7). Furthermore, it will be shown that the computation of the first obstruction to HNP corresponding to an extension can be reduced to the computation in finite group theory, and the computation with Galois groups of local extensions at a finite number of primes of $k$ (Theorem 1.11). Combining Theorem 3 of [12, p. 343] and Theorem 1.11 we obtain quite a deep result about the equality of norm groups as subgroups of the multiplicative group of the base field (Theorem 1.12). In the second section we apply Theorem 1.12 to investigate the equality of norm groups and, in particular, solitary extensions of algebraic number fields. Solitary extensions were initially introduced in [7], and the Galois solitary extensions were investigated extensively. In the present work we investigate solitary non-Galois extensions of low degrees. We note that many results in the second section are obtained using the results from [3] on the first obstructions to HNP corresponding to towers with Galois extensions. However, the more general notion of the first obstruction introduced above is required to investigate the equality of norm groups corresponding to a pair of extensions of degree six. In [12] we constructed two extensions $K/k$ and $L/k$ of degree six with the Galois groups of their normal closures isomorphic to $A_4$, and such that $N(K/k) = N(L/k)$ and $i(K/k) = i(L/k) = 2$. Quite involved computation in [12] shows that $N_{K/k}K^* \neq N_{L/k}L^*$. The results obtained in [3] on the first obstructions corresponding to towers are not applicable here to prove the above inequality of norm groups (see the paragraph preceding Proposition 2.5). Using the results on the first obstructions to HNP corresponding to extensions we establish a criterion for the equality of norm groups corresponding to extensions of degree six (Proposition 2.5). In particular, it follows by Proposition 2.5 that $N_{K/k}K^* \neq N_{L/k}L^*$ in the above mentioned example.

## 1. AN OBSTRUCTION TO HNP

We begin by mentioning two group isomorphisms that will be used in the future.

LEMMA 1.1. *Let $A$, $B$, $C$, and $D$ be groups, and let*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \big\downarrow \alpha & & \big\downarrow \beta \\ C & \xrightarrow{\psi} & D, \end{array}$$

*be a commutative diagram of group homomorphisms with a surjective homomorphism $\varphi$. Then*

$$[\operatorname{Ker}\psi \cap \alpha(A)]/\alpha(\operatorname{Ker}\varphi) \cong \operatorname{Ker}\beta/\varphi(\operatorname{Ker}\alpha).$$

*Proof.* We define a mapping

$$\chi\colon \operatorname{Ker}\psi \cap \alpha(A) \to \operatorname{Ker}\beta/\varphi(\operatorname{Ker}\alpha)$$

by the rule $\chi(\alpha(a)) = \varphi(a)\,\varphi(\operatorname{Ker}\alpha)$. It is easy to show that $\chi$ is a well defined surjective homomorphism, and $\operatorname{Ker}\chi = \alpha(\operatorname{Ker}\varphi)$. ∎

Let $L/k$ and $T/k$ be finite extensions of an algebraic number field $k$. Let $E/k$ be an arbitrary finite Galois extension of $k$ containing $L$ and $T$. Consider the following commutative diagram

$$\begin{tikzcd} J_L/N_{E/L}J_E \arrow[r, "\varphi"] \arrow[d, "\alpha"] & J_L/(N_{E/L}J_E)\,L^* \arrow[d, "\beta"] \\ J_k/N_{T/k}J_T \arrow[r, "\psi"] & J_k/(N_{T/k}J_T)\,k^*, \end{tikzcd} \tag{1}$$

where $\varphi$ and $\psi$ are the canonical surjective homomorphisms, and $\alpha$, $\beta$ are induced by the norm operator $N_{L/k}$. We mention the following well known isomorphism for future reference.

*Remark.* Let $B \subseteq A$ and $X$ be subgroups of an Abelian group. Then the kernel of the natural homomorphism $A \to AX/BX$ coincides with $A \cap BX = (A \cap X)B$. We thus obtain an isomorphism

$$AX/BX \cong A/(A \cap X)B.$$

PROPOSITION 1.2. *Let $L$ and $T$ be finite extensions of an algebraic number field $k$. Let $E$ be a finite Galois extension of $k$ containing $L$ and $T$. Then in the notation of the diagram* (1) *the first obstruction to HNP for $L/k$ corresponding to $T/k$ is isomorphic to* $\operatorname{Ker}\beta/\varphi(\operatorname{Ker}\alpha)$.

*Proof.* We note that $\operatorname{Ker}\psi \cap \alpha(J_L/N_{E/L}J_E)$ is equal to

$$\frac{(N_{T/k}J_T)k^*}{N_{T/k}J_T} \cap \frac{N_{L/k}J_L N_{T/k}J_T}{N_{T/k}J_T} = \frac{[k^* \cap N_{L/k}J_L N_{T/k}J_T]\,N_{T/k}J_T}{N_{T/k}J_T},$$

and $\alpha(\operatorname{Ker}\varphi)$ is equal to

$$\alpha\left[\frac{(N_{E/L}J_E)\,L^*}{N_{E/L}J_E}\right] = \frac{N_{L/k}L^* N_{T/k}J_T}{N_{T/k}J_T}.$$

So by the above remark the factor group of $\operatorname{Ker}\psi \cap \alpha(J_L/N_{E/L}J_E)$ by $\alpha(\operatorname{Ker}\varphi)$ is isomorphic to

$$\frac{[k^* \cap N_{L/k}J_L N_{T/k}J_T]\, N_{T/k}J_T}{N_{L/k}L^* N_{T/k}J_T} \cong \frac{k^* \cap N_{L/k}J_L N_{T/k}J_T}{N(T/k)\, N_{L/k}L^*}.$$

By Lemma 1.1 the first obstruction to HNP for $L/k$ corresponding to $T/k$ is isomorphic to $\operatorname{Ker}\beta/\varphi(\operatorname{Ker}\alpha)$. ∎

We wish to determine for an arbitrary pair of finite extensions of algebraic number fields $K/k$ and $L/k$ whether the equality $N_{K/k}K^* = N_{L/k}L^*$ holds. For the equality to hold it is necessary, but not sufficient, that $N_{K/k}J_K = N_{L/k}J_L$. The last equality has a group theoretic interpretation [12, Theorem 3, p. 343]. Indeed, for any subgroup $H$ of a finite group $G$ we define

$$\mathscr{P}_G(H) = \{g^{-1}hg \,/\, h \in H \text{ of prime power order, } g \in G\}.$$

Let $E/k$ be a finite Galois extension containing $K$ and $L$. We set $G = G(E/k)$, $H = G(E/K)$, and $N = G(E/L)$. By Theorem 3 of [12, p. 343] the inclusion $N_{K/k}J_K \subseteq N_{L/k}J_L$ is equivalent to the inclusion $\mathscr{P}_G(H) \subseteq \mathscr{P}_G(N)$. In particular, $N_{K/k}J_K = N_{L/k}J_L$ iff $\mathscr{P}_G(H) = \mathscr{P}_G(N)$. So if $\mathscr{P}_G(H) = \mathscr{P}_G(N)$, then $k^* \cap N_{K/k}J_K N_{T/k}J_T = k^* \cap N_{L/k}J_L N_{T/k}J_T$ for any finite extension $T/k$. Also, the equality $N_{K/k}K^* = N_{L/k}L^*$ obviously implies that $N(T/k)\, N_{K/k}K^* = N(T/k)\, N_{L/k}L^*$. It follows that the equality of norm groups implies the equality of the first obstructions to HNP for $K/k$ and $L/k$, respectively, corresponding to an arbitrary finite extension $T/k$. In particular, these two obstructions are isomorphic. Although the following two conditions—the group theoretic condition, and the isomorphism of the first obstructions corresponding to an arbitrary finite extension $T/k$—seem to be quite strong to imply the equality $N_{K/k}K^* = N_{L/k}L^*$, it is still an open question whether these conditions are sufficient. We will describe the factor group of $\operatorname{Ker}\beta$ by $\varphi(\operatorname{Ker}\alpha)$ in terms of Galois groups of local extensions. This in turn will yield by Proposition 1.2 a description of the first obstruction to HNP corresponding to a finite extension in terms of Galois groups of local extensions. However, we will first make an observation concerning the total obstructions to HNP. Let $L/k$ and $T/k$ be finite extensions of algebraic number fields. We have the following tower of groups

$$N(T/k)\, N_{L/k}L^* \subseteq N(T/k)\, N(L/k) \subseteq k^* \cap N_{L/k}J_L N_{T/k}J_T.$$

By the remark before Proposition 1.2 the factor group of the second group by the first group in this tower is isomorphic to the factor group of $N(L/k)$

by $[N(L/k) \cap N(T/k)]\, N_{L/k} L^*$. We thus obtain, in the notation of diagram (1), the following short exact sequence of finite groups

$$\frac{N(L/k)}{[N(L/k) \cap N(T/k)]\, N_{L/k} L^*} \rightarrowtail \frac{\operatorname{Ker} \beta}{\varphi(\operatorname{Ker} \alpha)} \twoheadrightarrow \frac{k^* \cap N_{L/k} J_L N_{T/k} J_T}{N(L/k)\, N(T/k)}. \tag{2}$$

Let $X$ be the pull-back of the canonical homomorphisms $N_{L/k} J_L \to C_k \leftarrow N_{T/k} J_T$. The kernel of the canonical homomorphism $\theta$: $X \to C_k$, induced by the above homomorphisms, coincides with $N(L/k) \times N(T/k)$. The image of $\theta$ is equal to $N_{L/k} C_L \cap N_{T/k} C_T$. By global class field theory $\operatorname{Im} \theta = N_{F/k} C_F$, where $F$ is the compositum of the maximal Abelian extensions of $k$ contained in $L$ and $T$, respectively. It follows that $\theta$ induces an isomorphism $\bar{\theta}$ from $X/\operatorname{Ker}\theta$ onto $N_{F/k} C_F$. On the other hand for $X$, being a subgroup of the direct product $N_{L/k} J_L \times N_{T/k} J_T$, there is a natural homomorphism $\delta$ from $X$ onto the factor group of $k^* \cap N_{L/k} J_L N_{T/k} J_T$ by $N(L/k)\, N(T/k)$. This epimorphism is given by $\delta(x, y) = xy^{-1} N(L/k)\, N(T/k)$. Since $\operatorname{Ker} \theta \subseteq \operatorname{Ker} \delta$, it follows that $\delta$ induces an epimorphism $\bar{\delta}$ from $X/\operatorname{Ker} \theta$ onto the factor group of $k^* \cap N_{L/k} J_L N_{T/k} J_T$ by $N(L/k)\, N(T/k)$. We thus obtain that this factor group is a homomorphic image of $N_{F/k} C_F$ under $\bar{\delta} \circ \bar{\theta}^{-1}$. Since the above factor group is finite, there exits a finite Abelian extension $R/k$ containing $F$ such that $N_{R/k} C_R = \operatorname{Ker}(\bar{\delta} \circ \bar{\theta}^{-1})$. Now if $E$ is a finite Galois extension of $k$ containing $L$ and $T$, and $G = G(E/k)$, $H = G(E/L)$, $N = G(E/T)$, then $F$ is the fixed field of $HG' \cap NG'$, where $G' = [G, G]$ is the commutator subgroup of $G$. We wish to show that $N_{E'/k} C_{E'} \subseteq N_{R/k} C_R$, where $E'$ is the maximal Abelian extension of $k$ contained in $E$. Indeed, let $\alpha k^* \in N_{E'/k} C_{E'}$ be an arbitrary element. Since $(N_{E'/k} J_{E'})\, k^* = (N_{E/k} J_E)\, k^*$, we may assume that $\alpha \in N_{E/k} J_E$. So $\alpha$ belongs to the intersection of $N_{L/k} J_L$ with $N_{T/k} J_T$, and therefore $\bar{\theta}^{-1}(\alpha k^*) = (\alpha, \alpha) \operatorname{Ker} \theta$. It follows that $\alpha k^* \in \operatorname{Ker}(\bar{\delta} \circ \bar{\theta}^{-1})$. By global class field theory $R \subseteq E'$, and $G(R/F)$ is isomorphic to the factor group of $k^* \cap N_{L/k} J_L N_{T/k} J_T$ by $N(L/k)\, N(T/k)$. We thus obtain by (2) the following proposition.

PROPOSITION 1.3. *Let $L/k$ and $T/k$ be finite extensions of algebraic number fields. Let $E$ be a finite Galois extension of $k$ containing $L$ and $T$. We set $G = G(E/k)$, $H = G(E/L)$, $N = G(E/T)$. Then the factor group of $k^* \cap N_{L/k} J_L N_{T/k} J_T$ by $N(L/k)\, N(T/k)$ is a homomorphic image of $HG' \cap NG'/G'$, where $G'$ is the commutator subgroup of $G$. In particular, if $HG' \cap NG' = G'$, then in the notation of diagram* (1)

$$\frac{N(L/k)}{[N(L/k) \cap N(T/k)]\, N_{L/k} L^*} \cong \frac{\operatorname{Ker} \beta}{\varphi(\operatorname{Ker} \alpha)}.$$

*Moreover, if* $\mathscr{P}_G(N) \subseteq \mathscr{P}_G(H)$, *then*

$$N(L/k)/N(T/k)\, N_{L/k}L^* \cong \operatorname{Ker}\beta/\varphi(\operatorname{Ker}\alpha).$$

A group $G$ which coincides with its commutator subgroup $G'$ is called a *perfect* group. Of course, any non-Abelian simple group is perfect.

COROLLARY 1.4. *Let $L$ be a finite extension of an algebraic number field $k$ such that the Galois group of its normal closure $E$ over $k$ is perfect. Then for any field $k \subseteq T \subseteq E$ in the notation of the diagram* (1)

$$\frac{N(L/k)}{[N(L/k) \cap N(T/k)]\, N_{L/k}L^*} \cong \frac{\operatorname{Ker}\beta}{\varphi(\operatorname{Ker}\alpha)}.$$

We wish now to describe the factor group $\operatorname{Ker}\beta/\varphi(\operatorname{Ker}\alpha)$ in terms of Galois groups of local extensions. Let $X/Y$ be a finite extension (not necessarily Galois) of algebraic number fields. In [11] we defined

$$[N_{X/Y}X^*]_\upsilon = \prod_{\nu \mid \upsilon} N_{X_\nu/Y_\upsilon} X_\nu^*$$

for each prime $\upsilon$ of $Y$. It follows by Proposition 1.2 (see [11, p. 112]) that $[N_{X/Y}X^*]_\upsilon$ is the topological closure of $N_{X/Y}X^*$ in $Y_\upsilon^*$.

PROPOSITION 1.5. *Let $X/Y$ be a finite extension* (*not necessarily Galois*) *of algebraic number fields. Then the canonical mapping* (*into a direct sum*)

$$J_Y/N_{X/Y}J_X \to \coprod_\upsilon Y_\upsilon^*/[N_{X/Y}X^*]_\upsilon$$

*given by* $\alpha N_{X/Y}J_X \mapsto (\alpha_\upsilon[N_{X/Y}X^*]_\upsilon)_\upsilon$ *is an isomorphism onto, where $\upsilon$ ranges over all primes of $Y$.*

*Proof.* We define a homomorphism

$$\varphi\colon J_Y \to \coprod_\upsilon Y_\upsilon^*/[N_{X/Y}X^*]_\upsilon$$

as $\varphi(\alpha) = (\alpha_\upsilon[N_{X/Y}X^*]_\upsilon)_\upsilon$. We need to show that $\alpha_\upsilon \in [N_{X/Y}X^*]_\upsilon$ for almost all primes $\upsilon$ of $Y$. Indeed, almost all primes $\upsilon$ of $Y$ are unramified in $X$. So $[N_{X/Y}X^*]_\upsilon = \prod_{\nu \mid \upsilon} N_{X_\nu/Y_\upsilon} X_\nu^*$ contains the unit group $U_{Y_\upsilon}$ of $Y_\upsilon$ for almost all primes $\upsilon$ of $Y$. On the other hand $\alpha_\upsilon \in U_{Y_\upsilon}$ for almost all primes $\upsilon$ of $Y$. So $\alpha_\upsilon \in [N_{X/Y}X^*]_\upsilon$ for almost all primes $\upsilon$ of $Y$. Also, since almost all components of an element of $\coprod_\upsilon Y_\upsilon^*/[N_{X/Y}X^*]_\upsilon$ are equal to 1, it follows that $\varphi$ is an epimorphism. To complete the proof of the proposition it remains to show that $\operatorname{Ker}\varphi = N_{X/Y}J_X$. Let $N_{X/Y}(\beta) \in N_{X/Y}J_X$ be an arbitrary element. Then for each prime $\upsilon$ of $Y$, $N_{X/Y}(\beta)_\upsilon = \prod_{\nu|\upsilon} N_{X_\nu/Y_\upsilon}(\beta_\nu) \in [N_{X/Y}X^*]_\upsilon$. So $N_{X/Y}(\beta) \in \operatorname{Ker}\varphi$. To prove the inclusion in the opposite direction we assume that $\delta \in \operatorname{Ker}\varphi$ is an arbitrary idele of $Y$, i.e., $\delta_\upsilon \in [N_{X/Y}X^*]_\upsilon$ for each prime $\upsilon$ of $Y$. We will show that there is an idele $\gamma \in J_X$ such that $\delta = N_{X/Y}(\gamma)$. Let $S$ be the set of primes $\upsilon$ of $Y$ for which at least one of the following conditions is satisfied: $\upsilon$ is an infinite prime, or $\delta_\upsilon$ is not a unit in $Y_\upsilon$, or $\upsilon$ is ramified in $X$. Let $\upsilon \notin S$ be an arbitrary prime of $Y$. Let $\nu_1, ..., \nu_m$ be the primes of $X$ which divide $\upsilon$. Since $\upsilon$ is unramified in $X$, it follows that $X_{\nu_1}/Y_\upsilon$ is an unramified extension, and therefore there is a unit $\gamma_{\nu_1}$ such that $\delta_\upsilon = N_{X_{\nu_1}/Y_\upsilon}(\gamma_{\nu_1})$. We set $\gamma_{\nu_2} = \cdots = \gamma_{\nu_m} = 1$. Now let $\upsilon \in S$ be an arbitrary element. Since $\delta_\upsilon \in [N_{X/Y}X^*]_\upsilon = \prod_{\nu|\upsilon} N_{X_\nu/Y_\upsilon} X_\nu^*$, it follows that for each $\nu|\upsilon$ there is $\gamma_\nu \in X_\nu^*$ such that $\delta_\upsilon = \prod_{\nu|\upsilon} N_{X_\nu/Y_\upsilon}(\gamma_\nu)$. Since $S$ is a finite set we can define an idele $\gamma \in J_X$ such that $(\gamma)_\nu = \gamma_\nu$ for all primes $\nu$ of $X$. So $(N_{X/Y}(\gamma))_\upsilon = \prod_{\nu|\upsilon} N_{X_\nu/Y_\upsilon}(\gamma_\nu) = \delta_\upsilon$ for each prime $\upsilon$ of $Y$, and therefore $\delta = N_{X/Y}(\gamma) \in N_{X/Y}J_X$. ∎

We note that if $X/Y$ is a finite Galois extension, then $[N_{X/Y}X^*]_\upsilon = N_{X_\upsilon/Y_\upsilon}X_\upsilon^*$. It follows now by local class field theory that in the case of Galois extensions $[N_{X/Y}X^*]_\upsilon = N_{X'_\upsilon/Y_\upsilon}X_\upsilon'^*$, where $X'_\upsilon$ is the maximal Abelian extension of $Y_\upsilon$ contained in $X_\upsilon$. In the general case we set

$$X'_\upsilon = \bigcap_{\nu|\upsilon} X'_\nu, \tag{3}$$

for each prime $\upsilon$ of $Y$, where $X'_\nu$ is the maximal Abelian extension of $Y_\upsilon$ contained in $X_\nu$. By local class field theory

$$[N_{X/Y}X^*]_\upsilon = \prod_{\nu|\upsilon} N_{X'_\nu/Y_\upsilon} X_\nu'^* = N_{X'_\upsilon/Y_\upsilon} X_\upsilon'^*.$$

By Proposition 1.5 we thus obtain the following corollary.

Corollary 1.6. *Let $X/Y$ be a finite extension (not necessarily Galois) of algebraic number fields. Then the canonical mapping*

$$J_Y/N_{X/Y}J_X \to \coprod_\upsilon Y_\upsilon^*/N_{X'_\upsilon/Y_\upsilon}X_\upsilon'^*$$

*given by $\alpha N_{X/Y}J_X \mapsto (\alpha_\upsilon N_{X'_\upsilon/Y_\upsilon}X_\upsilon'^*)_\upsilon$ is an isomorphism onto, where $\upsilon$ ranges over all primes of $Y$, and $X'_\upsilon$ is defined in (3).*

Let $L/k$ and $T/k$ be finite extensions of algebraic number fields. Let $E/k$ be a finite Galois extension containing $L$ and $T$, and suppose that $G = G(E/k)$, $H = G(E/L)$, and $N = G(E/T)$. For each prime $\upsilon$ of $k$ we fix a $k$-embedding of $E$ into the algebraic closure $\tilde{k}_\upsilon$ of the completion $k_\upsilon$ of $k$

at $\upsilon$. This will also fix a decomposition group $G_\upsilon = \mathrm{res}_{E_\upsilon/E}[G(E_\upsilon/k_\upsilon)]$ of $\upsilon$ in $E$. There is a one-to-one correspondence between the primes $\omega$ of $L$ above $\upsilon$ and the distinct double cosets $Hx_\omega G_\upsilon$ of $H$ and $G_\upsilon$ in $G$. Furthermore, $H_\omega = H \cap x_\omega G_\upsilon x_\omega^{-1}$ is a decomposition group of $\omega$ in $E$. Similarly, there is a one-to-one correspondence between the primes $\nu$ of $T$ above $\upsilon$ and the distinct double cosets $Ny_\nu G_\upsilon$ of $N$ and $G_\upsilon$ in $G$, and $N_\nu = N \cap y_\nu G_\upsilon y_\nu^{-1}$ is a decomposition group of $\nu$ in $E$. Let $\upsilon$ be an arbitrary prime of $k$. Suppose that $G = \bigcup_{\omega|\upsilon} Hx_\omega G_\upsilon = \bigcup_{\nu|\upsilon} Ny_\nu G_\upsilon$ are decompositions of $G$ into the unions of distinct double cosets of $H$, $G_\upsilon$ and $N$, $G_\upsilon$ in $G$, respectively. By local class field theory

$$\prod_{\omega|\upsilon} L_\omega^*/N_{E_\upsilon/L_\omega} E_\upsilon^* \cong \prod_{\omega|\upsilon} H_\omega/H'_\omega, \tag{4}$$

where $H'_\omega$ is the commutator subgroup of $H_\omega$. It follows by the definition of $T'_\upsilon$ that $T'_\upsilon$ is the fixed field of $\prod_{\nu|\upsilon} G(E_\upsilon/T_\nu)\, G(E_\upsilon/k_\upsilon)'$, where $G(E_\upsilon/k_\upsilon)'$ is the commutator subgroup of $G(E_\upsilon/k_\upsilon)$. By local class field theory

$$k_\upsilon^*/N_{T'_\upsilon/k_\upsilon} T_\upsilon'^* \cong G_\upsilon \Big/ \prod_{\nu|\upsilon} (N_\nu^{y_\nu} G'_\upsilon), \tag{5}$$

where $N_\nu^{y_\nu} = y_\nu^{-1} N_\nu\, y_\nu$ and $G'_\upsilon = [G_\upsilon, G_\upsilon]$. We thus obtain by Corollary 1.6 and (4), (5) that

$$J_L/N_{E/L} J_E \cong \coprod_\upsilon \Big( \prod_{\omega|\upsilon} H_\omega/H'_\omega \Big) \qquad \text{and} \qquad J_k/N_{T/k} J_T \cong \coprod_\upsilon G_\upsilon \Big/ \prod_{\nu|\upsilon} (N_\nu^{y_\nu} G'_\upsilon), \tag{6}$$

where $\upsilon$ ranges over all primes of $k$. The factor groups in the right column in the diagram (1) are canonically isomorphic to $C_L/N_{E/L} C_E$ and $C_k/N_{T/k} C_T$, respectively. We expand diagram (1) into two commutative diagrams as

$$\begin{tikzcd} J_L/N_{E/L} J_E \arrow[r, "\varphi"] \arrow[d, "\alpha_1"] & C_L/N_{E/L} C_E \arrow[d, "\beta_1"] \\ J_k/N_{E/k} J_E \arrow[r, "\chi"] & C_k/N_{E/k} C_E, \end{tikzcd} \tag{7}$$

where $\chi$ is the canonical homomorphism, and $\alpha_1, \beta_1$ are induced by the norm operators $N_{L/k}$ on $J_L$ and $C_L$, respectively. The mappings in the second diagram are all canonical homomorphisms.

$$\begin{tikzcd} J_k/N_{E/k}J_E \arrow[r, "\chi"] \arrow[d, "\alpha_2"] & C_k/N_{E/k}C_E \arrow[d, "\beta_2"] \\ J_k/N_{T/k}J_T \arrow[r, "\psi"] & C_k/N_{T/k}C_T \end{tikzcd} \tag{8}$$

By [3] diagram (7) is equivalent to the commutative diagram

$$\begin{tikzcd} \coprod_{\upsilon}\left(\prod_{\omega\mid\upsilon} H_\omega/H'_\omega\right) \arrow[r, "\varphi'"] \arrow[d, "\prod_\upsilon \psi_\upsilon^H"] & H/H' \arrow[d, "\pi_1"] \\ \coprod_{\upsilon} G_\upsilon/G'_\upsilon \arrow[r, "\chi'"] & G/G' \end{tikzcd} \tag{9}$$

($\upsilon$ ranges over all primes of $k$), where $\varphi'$, $\chi'$, $\pi_1$ are canonical homomorphisms, and $\psi_\upsilon^H\colon \prod_{\omega\mid\upsilon} H_\omega/H'_\omega \to G_\upsilon/G'_\upsilon$ is defined by the rule

$$\psi_\upsilon^H[(h_\omega H'_\omega)_{\omega\mid\upsilon}] = \left(\prod_{\omega\mid\upsilon} x_\omega^{-1} h_\omega x_\omega\right) G'_\upsilon.$$

Indeed, $\beta_1$ is equivalent to $\pi_1$ by global class field theory. The equivalence of $\varphi$ and $\chi$ to $\varphi'$ and $\chi'$, respectively, follows from the fact that the global reciprocity isomorphism is equal to the product of the local ones. The equivalence of $\alpha_1$ and $\prod \psi_\upsilon^H$ is shown in details in [7, p. 8]. Similarly, by (6) we can show that diagram (8) is equivalent to the following commutative diagram with canonical homomorphisms.

$$\begin{tikzcd} \coprod_{\upsilon} G_\upsilon/G'_\upsilon \arrow[r, "\chi'"] \arrow[d, "\prod_\upsilon \psi_{N,\upsilon}"] & G/G' \arrow[d, "\pi_2"] \\ \coprod_{\upsilon} G_\upsilon \Big/ \prod_{\nu\mid\upsilon} (N_\nu^{y_\nu} G'_\upsilon) \arrow[r, "\psi'"] & G/NG'. \end{tikzcd} \tag{10}$$

Combining diagrams (9) and (10), and renaming $\varphi'$ to $\lambda$ we obtain a commutative diagram which is equivalent to the diagram (1),

$$\begin{tikzcd} \coprod_{\upsilon}\left(\prod_{\omega\mid\upsilon} H_\omega/H'_\omega\right) \arrow[r, "\lambda"] \arrow[d, "\prod_\upsilon \psi_\upsilon"] & H/H' \arrow[d, "\pi"] \\ \coprod_{\upsilon} G_\upsilon \Big/ \prod_{\nu\mid\upsilon} (N_\nu^{y_\nu} G'_\upsilon) \arrow[r, "\psi'"] & G/NG' \end{tikzcd} \tag{11}$$

($\upsilon$ ranges over all primes of $k$), where $\pi = \pi_2 \circ \pi_1$, and $\psi_\upsilon = \psi_{N,\upsilon} \circ \psi_\upsilon^H$ is defined by

$$\psi_\upsilon[(h_\omega H'_\omega)_{\omega|\upsilon}] = \left( \prod_{\omega|\upsilon} x_\omega^{-1} h_\omega x_\omega \right) \prod_{\nu|\upsilon} (N_\nu^{y_\nu} G'_\upsilon). \tag{12}$$

By Proposition 1.2 we thus obtain the following theorem.

THEOREM 1.7. *Let $L/k$ and $T/k$ be finite extensions of an algebraic number field $k$. Let $E/k$ be a finite Galois extension containing $L$ and $T$. We set $G = G(E/k)$, $H = G(E/L)$, and $N = G(E/T)$. For each prime $\upsilon$ of $k$ let $G_\upsilon$ be a decomposition group of $\upsilon$ in $E$. Suppose that $G = \bigcup_{\omega|\upsilon} Hx_\omega G_\upsilon = \bigcup_{\nu|\upsilon} Ny_\nu G_\upsilon$ are the decompositions of $G$ into the unions of distinct double cosets. Let $H_\omega = H \cap x_\omega G_\upsilon x_\omega^{-1}$ and $N_\nu = N \cap y_\nu G_\upsilon y_\nu^{-1}$ be decomposition groups, respectively, of primes $\omega$ of $L$ and $\nu$ of $T$ above $\upsilon$ in $E$. Then in the notation of the diagram (11) the first obstruction to HNP for $L/k$ corresponding to the extension $T/k$ is isomorphic to* $\operatorname{Ker} \pi/\lambda[\operatorname{Ker} \prod_\upsilon \psi_\upsilon]$.

*Remark.* We note that $\operatorname{Ker} \pi = (H \cap NG')/H'$. Let $\lambda_\upsilon$ be the restriction of $\lambda$ to $\prod_{\omega|\upsilon} H_\omega/H'_\omega$ for each prime $\upsilon$ of $k$. Then $\lambda[\operatorname{Ker} \prod_\upsilon \psi_\upsilon]$ is the subgroup of $(H \cap NG')/H'$ generated by the subgroups $\lambda_\upsilon[\operatorname{Ker} \psi_\upsilon]$, where $\upsilon$ ranges over all primes of $k$.

Let $V$ be the set of all primes $\upsilon$ of $k$ for which $G_\upsilon$ is a cyclic group. We wish to describe the group generated by $\bigcup_{\upsilon \in V} \lambda_\upsilon[\operatorname{Ker} \psi_\upsilon]$. We note first that if $G_\upsilon = C$ is a cyclic group, then $\prod_{\nu|\upsilon} (N_\nu^{y_\nu} G'_\upsilon) = \prod_{g \in G} (N^g \cap C)$. Indeed, let $g \in G = \bigcup_{\nu|\upsilon} Ny_\nu C$ be an arbitrary element. Then there is $\nu|\upsilon$ such that $g = xy_\nu c$ for some $x \in N$ and $c \in C$.

$$N^g \cap C = N^{xy_\nu c} \cap C = N^{y_\nu c} \cap C = (N^{y_\nu} \cap C)^c = N^{y_\nu} \cap C.$$

So

$$\prod_{\nu|\upsilon} (N_\nu^{y_\nu} G'_\upsilon) = \prod_{\nu|\upsilon} (N \cap y_\nu C y_\nu^{-1})^{y_\nu} = \prod_{\nu|\upsilon} (N^{y_\nu} \cap C) = \prod_{g \in G} (N^g \cap C).$$

We set $N_C = \prod_{g \in G} (N^g \cap C)$ for each cyclic subgroup $C$ of $G$. We define a set

$$G(C) = \{[h, g]x \,/\, h \in H \cap g(H \cap C)g^{-1}, x \in H \cap N_C, g \in G\} \tag{13}$$

for each cyclic subgroup $C$ of $G$, where $[h, g] = h^{-1}g^{-1}hg$. Then $H' \subseteq \bigcup_C G(C) \subseteq H \cap NG'$ ($H'$ and $G'$ are commutator subgroups of $H$ and $G$, respectively), where the union is taken over all cyclic subgroups $C$ of $G$. Indeed, to prove the first inclusion we assume that $h_1, h_2 \in H$ are arbitrary elements. Let $C = \langle h_2^{-1} h_1 h_2 \rangle$, $g = h_2$, $x = 1$, and $h = h_1$ in (13). Then

$[h_1, h_2] \in G(C)$. To prove the second inclusion we assume that $C$ is an arbitrary cyclic subgroup of $G$, and $[h, g]x \in G(C)$ is an arbitrary element. It follows that $[h, g]x \in H$. Furthermore, $[h, g]x = xx^{-1}[h, g]\, x \in xG'$. So to prove the second inclusion it remains to show that $x \in NG'$. Indeed, suppose that $G = \{g_1, ..., g_n\}$, and $x = \prod_{i=1}^{n} x_i^{g_i}$ for some $x_i^{g_i} \in N^{g_i} \cap C$. It follows that $x = \prod_{i=1}^{n} x_i^{g_i} = \prod_{i=1}^{n} x_i[x_i, g_i] \in NG'$. We now define a group generated by the union of $G(C)$.

Definition. $\Phi^G(H, N) = \langle \bigcup_C G(C) \rangle$, *where* $C$ *runs over all cyclic subgroups of* $G$.

We thus obtain that $H' \subseteq \Phi^G(H, N) \subseteq H \cap NG'$. In [3] a subgroup $H' \subseteq \Phi^G(H) \subseteq H \cap G'$ is defined as

$$\Phi^G(H) = \langle \{[h, g] \,/\, h \in H \cap gHg^{-1},\ g \in G\} \rangle.$$

It follows by the definition of $\Phi^G(H, N)$ that if $N = 1$, then $\Phi^G(H, N) = \Phi^G(H)$. The following theorem is a generalization of Theorem 2 of [3, p. 307] and its proof is a modification of the proof of that theorem.

Theorem 1.8. *In the notation of Theorem* 1.7 *and the diagram* (11), *let* $\lambda_v$ *be the restriction of* $\lambda$ *to* $\prod_{\omega | v} H_\omega / H'_\omega$ *for each prime* $v$ *of* $k$. *If* $V$ *is the set of primes* $v$ *of* $k$ *for which* $G_v$ *is a cyclic group, then*

$$\left\langle \bigcup_{v \in V} \lambda_v[\operatorname{Ker} \psi_v)] \right\rangle = \Phi^G(H, N)/H'. \tag{14}$$

*Proof.* We will first prove the inclusion $\supseteq$. Let $C$ be an arbitrary cyclic subgroup of $G$, and let $[h, g]x \in G(C)$ be an arbitrary element. We will consider two cases: $g \in HC$ and $g \notin HC$.

We assume first that $g \in HC$, and $g = h_1 c_1$. Since $h_1^{-1} g \in C$, it follows that $gh_1^{-1} gg^{-1} \in gCg^{-1}$, i.e., $gh_1^{-1} \in gCg^{-1}$. Also, $h \in gCg^{-1}$. So

$$[h, g]\,x = h^{-1} h_1^{-1}(h_1 g^{-1} hgh_1^{-1})\, h_1 x = [h, h_1]\, x \equiv x (\mathrm{mod}\ H').$$

So to prove the inclusion it suffices to show that $xH'$ belongs to the left side of (14). By Chebotarev's density theorem there is a prime $v$ of $k$ unramified in $E$ such that $G_v = C$. Let $G = HG_v \cup \cdots$ be the decomposition of $G$ into the union of the distinct double cosets of $H$ and $G_v$ in $G$. Let $\omega_1$ be the prime of $L$ above $v$ that corresponds to the double coset $HG_v$. Since $x \in H \cap C = H_{\omega_1}$, it follows that $\alpha = (x, 1, ..., 1) \in \prod_{\omega | v} H_\omega$. Also $x \in N_C$ implies $\psi_v(\alpha) = xN_C = 1$. So $\alpha \in \operatorname{Ker} \psi_v$, and therefore $xH' = \lambda_v(\alpha)$ belongs to the left side of (14).

We assume now that $g \notin HC$. There is a prime $v$ of $k$ for which $G_v = C$. Since $g \notin HC$, we have the following decomposition of $G$ into the union of the distinct double cosets with respect to $H$ and $G_v$: $G = HG_v \cup HgG_v \cup \cdots$. Let $\omega_1$ and $\omega_2$ be the primes of $L$ above $v$ corresponding to the double cosets $HG_v$ and $HgG_v$, respectively. Then $H_{\omega_1} = H \cap G_v$ and $H_{\omega_2} = H \cap gG_vg^{-1}$. Since $g^{-1}hgx \in H \cap G_v = H_{\omega_1}$ and $h \in H \cap gG_vg^{-1} = H_{\omega_2}$, it follows that $\alpha = (g^{-1}hgx, h^{-1}, 1, ..., 1) \in \prod_{\omega | v} H_\omega$. Also $\psi_v(\alpha) = (g^{-1}hgxg^{-1}h^{-1}g)N_C = xN_C$, since $g^{-1}hg \in C$ and $x \in C$. Now $x \in N_C$ implies $xN_C = 1$. So $\alpha \in \operatorname{Ker} \psi_v$. The following equality completes the proof of the inclusion.

$$\lambda_v(\alpha) = (g^{-1}hgxh^{-1})H' = (h^{-1}g^{-1}hgx)H' = [h, g]xH'.$$

We will now prove the inclusion $\subseteq$. Let $v \in V$ be an arbitrary element. We wish to show that $\lambda_v[\operatorname{Ker} \psi_v] \subseteq \Phi^G(H, N)/H'$. Let $G = \bigcup_{i=1}^n Hx_iG_v$ be the decomposition of $G$ into the union of the distinct double cosets with respect to $H$ and $G_v$. Then decomposition groups of the primes $\omega_i$ of $L$ above $v$ in $E$ are of the form $H_{\omega_i} = H \cap x_iG_vx_i^{-1}$ $(1 \leqslant i \leqslant n)$. We will prove that $\alpha = (h_1, ..., h_r, 1, ..., 1) \in \operatorname{Ker} \psi_v$ implies that $\lambda_v(\alpha) = (h_1 \cdot \cdots \cdot h_r)H' \in \Phi^G(H, N)/H'$ for each $1 \leqslant r \leqslant n$. We will prove this statement by induction on $r$.

For $r = 1$ we assume that $(h_1, 1, ..., 1) \in \operatorname{Ker} \psi_v$ is an arbitrary element. Then $x_1^{-1}h_1x_1 \in \prod_{g \in G} (N^g \cap G_v)$ (note that $G_v$ is a cyclic group). It follows that $h_1 \in \prod_{g \in G} (N^g \cap x_1G_vx_1^{-1})$. If $C = x_1G_vx_1^{-1}$, $g = 1$, $x = h_1$, $h = 1$ in (13), then $h_1 \in G(C)$, and therefore $h_1 \in \Phi^G(H, N)$.

We assume now that the above statement is true for all $r \leqslant s$, and we will prove that it is true for $r = s + 1$. Let $\alpha = (h_1, ..., h_{s+1}, 1, ..., 1) \in \operatorname{Ker} \psi_v$ be an arbitrary element. Let $G_v = \langle \sigma \rangle$. Since $h_i \in H_{\omega_i} = H \cap x_iG_vx_i^{-1}$, it follows that $h_i = x_i\sigma^{n_i}x_i^{-1}$ for some integers $n_i$. So $\alpha \in \operatorname{Ker} \psi_v$ implies that $\sigma$ to the exponent $\sum_{i=1}^{s+1} n_i$ is an element of $N_{G_v}$. Since $N_{G_v}$ is a subgroup of $G_v$, there is a generator $\sigma^t$ of $N_{G_v}$ such that $\sum_{i=1}^{s+1} n_i = tl$ for some integer $l$. Let $n_0 = \gcd(n_1, ..., n_{s+1}, tl)$. We set $m_i = n_i/n_0$ for each $i = 1, ..., s+1$. Then $\sum_{i=1}^{s+1} m_i = tl/n_0$ implies that $\gcd(m_1, ..., m_s, tl/n_0) = 1$. So there are integers $a_i$ $(1 \leqslant i \leqslant s)$ and $a$ such that $\sum_{i=1}^{s} m_ia_i + (tl/n_0)a = 1$. Let $\beta = (h_1^{m_{s+1}a_1}, ..., h_s^{m_{s+1}a_s}, h_{s+1}^{-1}, 1, ..., 1)$ be an element of $\prod_{i=1}^n H_{\omega_i}$.

$$\begin{aligned}
\psi_v(\beta) &= \left( \prod_{i=1}^{s} \sigma^{n_im_{s+1}a_i} \right) \sigma^{-n_{s+1}} N_{G_v} \\
&= \left( \prod_{i=1}^{s} \sigma^{n_im_{s+1}a_i} \right) \sigma^{-n_{s+1}(\sum_{i=1}^{s} m_ia_i + (tl/n_0)a)} N_{G_v} \\
&= \left( \prod_{i=1}^{s} \sigma^{n_im_{s+1}a_i - n_{s+1}m_ia_i} \right) \sigma^{-n_{s+1}(tl/n_0)a} N_{G_v}.
\end{aligned}$$

Since $n_im_{s+1} = n_i(n_{s+1}/n_0) = m_in_{s+1}$, and $n_{s+1}(tl/n_0) = (n_{s+1}/n_0)lt$, it follows that $\psi_v(\beta) = 1$, and therefore $\beta \in \operatorname{Ker} \psi_v$. We will now compute $\lambda_v(\beta)$.

$$\left(\prod_{i=1}^{s} h_i^{m_{s+1}a_i}\right) h_{s+1}^{-1} = \left(\prod_{i=1}^{s} h_i^{m_{s+1}a_i}\right) h_{s+1}^{-(\sum_{i=1}^{s} m_i a_i + (tl/n_0)a)}$$

$$= \left(\prod_{i=1}^{s} h_i^{m_{s+1}a_i}\right)\left(\prod_{i=1}^{s} h_{s+1}^{-m_i a_i}\right) h_{s+1}^{-(tl/n_0)a}$$

$$\equiv \left(\prod_{i=1}^{s} h_i^{m_{s+1}a_i} h_{s+1}^{-m_i a_i}\right) x_{s+1}\sigma^{(-m_{s+1}la)t} x_{s+1}^{-1} (\text{mod } H'). \tag{15}$$

Since $n_i m_{s+1} = n_{s+1} m_i$, it follows by setting $r_i = n_i m_{s+1} a_i$ ($i = 1, ..., s$) that

$$h_i^{m_{s+1}a_i} h_{s+1}^{-m_i a_i} = x_i \sigma^{r_i} x_i^{-1} x_{s+1} \sigma^{-r_i} x_{s+1}^{-1} = [x_i \sigma^{-r_i} x_i^{-1}, x_i x_{s+1}^{-1}] \in G(C),$$

where $C = x_{s+1} G_v x_{s+1}^{-1}$, $g = x_i x_{s+1}^{-1}$, $x = 1$, and $h = x_i \sigma^{-r_i} x_i^{-1}$ in (13). So the factor in the parentheses in (15) is an element of $\Phi^G(H, N)$. Now if we set $C = x_{s+1} G_v x_{s+1}^{-1}$, $g = 1$, $x = x_{s+1} \sigma^{(-m_{s+1} la)t} x_{s+1}^{-1}$, and $h = 1$ in (13), then $x_{s+1} \sigma^{-m_{s+1} lat} x_{s+1}^{-1} \in G(C)$. We thus obtain that $\lambda_v(\beta) \in \Phi^G(H, N)/H'$. We define $\gamma = \alpha\beta$. Since $\alpha, \beta \in \operatorname{Ker} \psi_v$, it follows that $\gamma \in \operatorname{Ker} \psi_v$. Also $\gamma$ has at most $s$ components different from 1. So by the induction hypothesis $\lambda_v(\gamma) \in \Phi^G(H, N)/H'$. Thus $\lambda_v(\alpha) \in \Phi^G(H, N)/H'$. ∎

The rather cumbersome definition of $\Phi^G(H, N)$ can be simplified as follows.

THEOREM 1.9. *Let $H$ and $N$ be subgroups of a finite group $G$. Then*

$$\Phi^G(H, N) = \Phi^G(H) \cdot \langle H \cap \mathscr{P}_G(N) \rangle.$$

To prove this theorem we will need the following lemma.

LEMMA 1.10. *Let $N$ be a subgroup of a finite group $G$. Suppose that $\mathscr{T}$ is the set of all cyclic subgroups of $G$ of prime power order. Then*

$$\bigcup_{C \in \mathscr{T}} N_C = \mathscr{P}_G(N),$$

*where $N_C = \prod_{g \in G} (N^g \cap C)$.*

*Proof.* Let $x \in \bigcup_{C \in \mathscr{T}} N_C$ be an arbitrary element. Suppose that $C \in \mathscr{T}$ is such that $x \in N_C$, and $N_C$, being a subgroup of $C$, is generated by $y$. For any $g \in G$ the inclusion $N^g \cap C \subseteq N_C = \langle y \rangle$ implies $N^g \cap C \subseteq N^g \cap \langle y \rangle$. So

$$\langle y \rangle = N_C = \prod_{g \in G} (N^g \cap C) \subseteq \prod_{g \in G} (N^g \cap \langle y \rangle) \subseteq \langle y \rangle.$$

We thus obtain that $\prod_{g \in G} (N^g \cap \langle y \rangle) = \langle y \rangle$. Since $y$ is of prime power order, it follows now by Lemma 1.6 of [11, p. 117] that there is $g \in G$ such that $g^{-1} y g \in N$. On the other hand $x = y^s$ for some integer $s$. So $g^{-1} x g \in N$, and therefore $x \in \mathcal{P}_G(N)$.

To prove the inclusion in the opposite direction, we assume that $z \in \mathcal{P}_G(N)$ is an arbitrary element. It follows that $z \in N^g$ for some $g \in G$, and $z$ is an element of prime power order. Applying one more time Lemma 1.6 [11] we conclude that $N_{\langle z \rangle} = \langle z \rangle$. So $z \in \bigcup_{C \in \mathcal{T}} N_C$. ∎

Let $X$ be a finite group, and let $p$ be a prime number. We denote by $X_p$ a Sylow $p$-subgroup of $X$. If a positive integer $n = p^k m$ with $(p, m) = 1$, then we define $n_p = p^k$.

*Proof of Theorem* 1.9. We define for each cyclic subgroup $C$ of $G$ the following two sets

$$A_C = \{ [h, g] \,/\, h \in H \cap g(H \cap C)\, g^{-1},\ g \in G \},$$

and $B_C = H \cap N_C$ (which is a subgroup of $C$). Let $\mathcal{S}$ be the set of all cyclic subgroups of $G$. It follows by the definition of $\Phi^G(H)$ that

$$\left\langle \bigcup_{C \in \mathcal{S}} A_C \right\rangle = \Phi^G(H). \tag{16}$$

We note that $A_C$ and $B_C$ are subsets of $G(C)$, and $G(C) = A_C \cdot B_C$ for any $C \in \mathcal{S}$. It follows that

$$\left\langle \bigcup_{C \in \mathcal{S}} G(C) \right\rangle = \left\langle \bigcup_{C \in \mathcal{S}} A_C \right\rangle \cdot \left\langle \bigcup_{C \in \mathcal{S}} B_C \right\rangle.$$

So by (16) we obtain

$$\Phi^G(H, N) = \left\langle \bigcup_{C \in \mathcal{S}} A_C \right\rangle \cdot \left\langle \bigcup_{C \in \mathcal{S}} B_C \right\rangle = \Phi^G(H) \cdot \left\langle \bigcup_{C \in \mathcal{S}} B_C \right\rangle. \tag{17}$$

Let $C$ be an arbitrary cyclic subgroup of $G$. Since $B_C$ is a subgroup of $C$, it follows that for any prime number $p$

$$(B_C)_p = H \cap N_C \cap C_p = H \cap (N_C)_p \tag{18}$$

(since $N_C$ is a subgroup of $C$, and $N_C \cap C_p = (N_C)_p$). Also, since $N^g \cap C$ is a subgroup of a cyclic group for any $g \in G$, it follows that

$$(N_C)_p = \left( \prod_{g \in G} (N^g \cap C) \right)_p = \prod_{g \in G} (N^g \cap C)_p = \prod_{g \in G} (N^g \cap C_p) = N_{C_p}.$$

So by (18)

$$(B_C)_p = H \cap (N_C)_p = H \cap N_{C_p} = B_{C_p}. \tag{19}$$

Let $\mathscr{T}$ be the set of all cyclic subgroups of $G$ of prime power order. We wish to show that

$$\left\langle \bigcup_{C \in \mathscr{S}} B_C \right\rangle = \left\langle \bigcup_{C \in \mathscr{T}} B_C \right\rangle. \tag{20}$$

The inclusion $\supseteq$ is obvious. To prove (20) it suffices, therefore, to show that $B_C \subseteq \langle \bigcup_{D \in \mathscr{T}} B_D \rangle$ for any cyclic subgroup $C$ of $G$. Indeed, by (19)

$$B_C = \prod_p (B_C)_p = \prod_p B_{C_p} \subseteq \left\langle \bigcup_{D \in \mathscr{T}} B_D \right\rangle,$$

where $p$ ranges over all prime divisors of the order of $B_C$. Now it follows by the definition of $B_D$ that

$$\bigcup_{D \in \mathscr{T}} B_D = H \cap \left( \bigcup_{D \in \mathscr{T}} N_D \right).$$

So by Lemma 1.10, and by (20) we obtain

$$\left\langle \bigcup_{C \in \mathscr{S}} B_C \right\rangle = \langle H \cap \mathscr{P}_G(N) \rangle.$$

The equality stated in Theorem 1.9 now follows by (17). ∎

In the notation of Theorem 1.7 and the remark following this theorem

$$\lambda \left[ \operatorname{Ker} \prod_v \psi_v \right] = \left\langle \bigcup_{v \in V} \lambda_v [\operatorname{Ker} \psi_v] \right\rangle \cdot \left\langle \bigcup_{v \in S} \lambda_v [\operatorname{Ker} \psi_v] \right\rangle,$$

where $V$ is the set of all primes $v$ of $k$ for which $G_v$ is a cyclic group, and $S$ is the complement of $V$ in the set of all primes of $k$. Of course, $S$ is a finite set of primes. By Theorem 1.8 the first factor in the above product has a group theoretic interpretation. This leads us to make the following definition.

DEFINITION. *Let $L/k$ and $T/k$ be finite extensions of algebraic number fields contained in a finite Galois extension $E/k$. Let $S$ be the set of primes of $k$ whose decomposition groups in $E$ are not cyclic. In the notation of the* *diagram* (11) *and the remark following Theorem* 1.7 *we define a subgroup* $H' \subseteq X_{L/k}(T, E) \subseteq H \cap NG'$ *as*

$$X_{L/k}(T, E)/H' = \prod_{v \in S} \lambda_v[\operatorname{Ker} \psi_v].$$

By Theorem 1.7 in conjunction with Theorems 1.8 and 1.9 we thus obtain the following theorem.

THEOREM 1.11. *Let $L/k$ and $T/k$ be finite extensions of algebraic number fields contained in a finite Galois extension $E/k$. We set $G = G(E/k)$, $H = G(E/L)$ and $N = G(E/T)$. Then the first obstruction to HNP for $L/k$ corresponding to $T/k$ is isomorphic to the factor group of $H \cap NG'$ by $\Phi^G(H)\langle H \cap \mathscr{P}_G(N)\rangle\, X_{L/k}(T, E)$.*

In the special case when $N = 1$ in Theorem 1.11 we obtain the results in [3] (Theorems 1 and 2). It follows that the factor group mentioned in Theorem 1.11 depends only on the extensions $L$ and $T$ of $k$. Theorem 3 of [12, p. 343] gives a group theoretic interpretation of the equality of norm groups of idele groups. Using this theorem and Theorem 1.11 we thus obtain the following necessary condition for the equality of norm groups as subgroups of the multiplicative group of the base field.

THEOREM 1.12. *Let $K/k$ and $L/k$ be finite extensions of algebraic number fields. If $N_{K/k}K^* = N_{L/k}L^*$, then for any finite extension $T/k$, and for any finite Galois extension $E/k$ containing $K$, $L$, and $T$ the following two conditions are satisfied: $\mathscr{P}_G(H_K) = \mathscr{P}_G(H_L)$ and*

$$\frac{H_K \cap NG'}{\Phi^G(H_K)\langle H_K \cap \mathscr{P}_G(N)\rangle\, X_{K/k}(T, E)} \cong \frac{H_L \cap NG'}{\Phi^G(H_L)\langle H_L \cap \mathscr{P}_G(N)\rangle\, X_{L/k}(T, E)}, \tag{21}$$

*where $G = G(E/k)$, $H_K = G(E/K)$, $H_L = G(E/L)$, and $N = G(E/T)$.*

*Remark.* The factor group in the left side of (21) depends only on the extensions $K$ and $T$ of $k$. So an arbitrary finite Galois extension of $k$ containing $K$ and $T$ can be used to compute the factor group in the left side of (21). Similarly, an arbitrary finite Galois extension of $k$ containing $L$ and $T$ can be used to compute the factor group in the right side of (21). On the other hand an arbitrary finite Galois extension of $k$ containing $K$ and $L$ can be used to verify the equality $\mathscr{P}_G(H_K) = \mathscr{P}_G(H_L)$.

## 2. EQUALITY OF NORM GROUPS AND SOLITARY EXTENSIONS

A finite extension $K$ of an algebraic number field $k$ is called *k-solitary* if for any finite extension $L/k$ the equality $N_{K/k}K^* = N_{L/k}L^*$ implies that $K$ and $L$ are conjugate over $k$ [7]. In [7] we mainly investigated solitary Galois extensions of algebraic number fields. By Theorem 2.7 of [7, p. 16] every solitary Galois extension is necessarily a 2-extension. In the case of Galois 2-extensions of degrees not exceeding 8 we established a criterion for an extension to be solitary [7, Theorem 4.8, p. 27]. We wish now to apply the results obtained in the first section to investigate when two finite extensions have equal norm groups. In particular, we wish to determine which non-Galois extensions of low degrees are solitary. We note first that the Galois group of the normal closure of a non-Galois extension of degree 4 is either $S_4$, $A_4$, or $D_8$ (the dihedral group of order 8).

PROPOSITION 2.1. *Let $K/k$ be a finite non-Galois extension of algebraic number fields. Let $G$ be the Galois group of the normal closure $\bar{K}$ of $K$ over $k$. Then $K$ is $k$-solitary in the following cases*: $(K:k)=3$; $(K:k)=4$, *and* $G=S_4$ *or* $G=A_4$. *Furthermore, if* $(K:k)=4$, $G=D_8$, *and $\bar{K}$ is $k$-solitary, then $K$ is $k$-solitary.*

*Proof.* Let $L/k$ be a finite extension such that $N_{K/k}K^* = N_{L/k}L^*$. Let $E$ be a finite Galois extension of $k$ containing $K$ and $L$. We set $\bar{G}=G(E/k)$, $H_K = G(E/K)$, and $H_L = G(E/L)$. By Theorem 3 of [12, p. 343] the equality of norm groups $N_{K/k}K^* = N_{L/k}L^*$ implies $\mathscr{P}_{\bar{G}}(H_K) = \mathscr{P}_{\bar{G}}(H_L)$. It follows by Theorem A of [6, p. 294] that a conjugate of $L$ over $k$ is contained in $K$ when $G=S_3$ (the case $(K:k)=3$) or $G=S_4$. Since $N_{L/k}L^* = N_{K/k}K^*$ is a proper subgroup of $k^*$ [11, Corollary 1.10, p. 120], it follows that $L \neq k$. Moreover, since $k$ is the only proper subfield of $K$ in both cases when $G=S_3$ or $G=S_4$, it follows that a conjugate of $L$ over $k$ coincides with $K$. So $K$ is $k$-solitary when $G=S_3$ or $G=S_4$.

We assume now that $G=A_4$. The equality $\mathscr{P}_{\bar{G}}(H_K) = \mathscr{P}_{\bar{G}}(H_L)$ implies $\mathscr{P}_G(\mathrm{res}_{E/\bar{K}}(H_K)) = \mathscr{P}_G(\mathrm{res}_{E/\bar{K}}(H_L))$. Since the only subgroups of $A_4$ whose order is divisible by 3 are the Sylow 3-subgroups and $A_4$, it follows that $K$ and $M = \bar{K} \cap L$ are conjugate over $k$. We wish to prove that $L = M$, which in turn will imply that $K$ is $k$-solitary. Suppose that $M$ is a proper subfield of $L$. Let $M \subset \tilde{L} \subseteq L$ be a minimal subfield of $L$ strictly containing $M$. Since $N_{K/k}K^* = N_{L/k}L^*$ and $N_{K/k}K^* = N_{M/k}M^*$, it follows that $N_{M/k}M^* = N_{\tilde{L}/k}\tilde{L}^*$. Let $\tilde{E}$ be the normal closure of $\tilde{L}$ over $k$. We note that $\tilde{E}$ contains $\bar{K}$. We set $\tilde{G} = G(\tilde{E}/k)$, $H = G(\tilde{E}/M)$, $N = G(\tilde{E}/\tilde{L})$, and $U = G(\tilde{E}/\bar{K})$. Let $A$ be a minimal normal subgroup of $\tilde{G}$ contained in $U$. Since the core $N_{\tilde{G}} = \bigcap_{g \in \tilde{G}} N^g = 1$, it follows that $A$ is not contained in $N$. On the other hand $N$ is a maximal subgroup of $H$. So $H = AN$. By Theorem 4.2 of [7, p. 21]

$A$ is an Abelian group, and therefore $A$ is an elementary Abelian $p$-group. Let $R = \tilde{G}/Z_{\tilde{G}}(A)$ and $\bar{H} = HZ_{\tilde{G}}(A)/Z_{\tilde{G}}(A)$, where $Z_{\tilde{G}}(A)$ is the centralizer of $A$ in $\tilde{G}$. Then $R$ is a group of automorphisms of $A$, and $\bar{H}$ is a subgroup of $R$ of index $2^d$ with $d \leqslant 2$. Let $B = A \cap N$. It follows that $B$ is a proper $\bar{H}$-invariant subgroup of $A$, $B_R = \bigcap_{r \in R} r(B) = 1$, and $B$ contains an $R$-conjugate of every element of $A$. By Lemma 4.5 of [7, p. 22] $A$ is a direct product of two copies of the cyclic group of order 3. Let $x \in H$ be an arbitrary element whose order is a power of 3, and $x \notin U$. It follows that the order of $xa$ is also a power of 3 for any $a \in A$. Now the equality of norm groups $N_{M/k} M^* = N_{\tilde{L}/k} \tilde{L}^*$ implies by Theorem 3 of [12, p. 343] that $\mathscr{P}_{\tilde{G}}(H) = \mathscr{P}_{\tilde{G}}(N)$. So there is $g \in \tilde{G}$ such that $(xa)^g \in N$. It follows that $x^g \in AN = H$. So $gU$ is an element of the normalizer in $\mathrm{A}_4 = \tilde{G}/U$ of the Sylow 3-subgroup $H/U = \langle xU \rangle$. Since the normalizer in $\mathrm{A}_4$ of any Sylow 3-subgroup coincides with the subgroup, it follows that $g \in H = AN$. We thus obtain that for any $a \in A$ there is $g \in A$ such that $(xa)^g \in N$. The action of $x$ on $A/B$ is trivial, since the order of $A/B$ is 3. So the commutator $[x, y]$ belongs to $B$ for any $y \in A$. For any $a \in A$ there is $g \in A$ such that $(xa)^g = g^{-1}xag = g^{-1}xga \in N$. On the other hand $g^{-1}xg = xb$ for some $b \in B$. So $xba = xab \in N$. Since $b \in N$, $xa \in N$ for any $a \in A$. It follows that $A$ is contained in $N$, contradiction. We thus obtain that $M = L$, and therefore $K$ is $k$-solitary.

Finally to prove the last statement in this proposition we assume that $(K\colon k) = 4$, $G = \mathrm{D}_8$, and $\bar{K}$ is $k$-solitary. Arguing as in the case $G = \mathrm{A}_4$, we obtain that $K$ and $M = \bar{K} \cap L$ are conjugate over $k$. To prove that $K$ is $k$-solitary it suffices to show that $M = L$. Suppose that $M$ is a proper subfield of $L$. Then $\bar{K}$ is a proper subfield of the compositum $\bar{K}L$. The equality $\mathscr{P}_{\bar{G}}(H_K) = \mathscr{P}_{\bar{G}}(H_L)$ implies $\mathscr{P}_{\bar{G}}(G(E/\bar{K}L)) = \mathscr{P}_{\bar{G}}(G(E/\bar{K}))$ (since $G(E/\bar{K})$ is normal in $\bar{G}$). By Theorem 4.7 of [7, p. 26] $G(E/\bar{K})$ contains a subgroup $X$ of index 2 such that $N_{\bar{G}}(X)/X$ contains a subgroup $Y$ which is isomorphic either to the cyclic group of order 8 or to the quaternion group of order 8. It follows that there is a field $k \subset T \subset \bar{K}$ such that $(\bar{K} : T) = 4$, and such that $F/T$ is a Galois extension with the Galois group isomorphic to $Y$, where $F$ is the fixed field of $X$. By Theorem 3.2 of [7, p. 17] $\bar{K}$ is not $k$-solitary, contradiction. So $M = L$, and $K$ is $k$-solitary. ∎

In [7, Section 5, pp. 28–31] we constructed a Galois extension $E$ of the field of rational numbers $\mathbf{Q}$ with the Galois group isomorphic to $\mathrm{D}_8$, and such that $E$ is $\mathbf{Q}$-solitary. By Proposition 2.1 a non-Galois extension $\mathbf{Q} \subset K \subset E$ of degree 4 over $\mathbf{Q}$ is $\mathbf{Q}$-solitary. We wish now to construct a non-Galois extension $K/k$ of degree 4 that is not solitary over $k$. By Proposition 2.1 the normal closure $\bar{K}$ of $K$ over $k$ must have the Galois group (over $k$) isomorphic to $\mathrm{D}_8$, and $\bar{K}$ is not $k$-solitary. Let $A = \langle a \rangle \times \langle b \rangle$ be a direct product of two cyclic groups of order 3. The natural action of $\mathrm{GL}(2, 3)$ on the additive group of a 2-dimensional vector space over

GF(3) induces an action of GL(2, 3) on $A$. Let $P$ be a Sylow 2-subgroup of GL(2, 3). It follows that $P$ has the presentation.

$$P = \langle x, y \,/\, x^8 = y^2 = 1,\ y^{-1}xy = x^3 \rangle.$$

The action of GL(2, 3) on $A$ defines an action of $P$ on $A$ as follows: $a^x = (ab)^2$, $b^x = ab^2$, $a^y = a^2$, and $b^y = b$. The action of $P$ on $A$ defines a semidirect product $G = AP$ of order 144. We note that the factor group of $P$ by its center $Z(P) = \langle x^4 \rangle$ is isomorphic to $\mathrm{D}_8$. Let $D$ be a subgroup of $P$ generated by $x^4$ and $y$. We define two subgroups of $G$: $H = AD$ and $N = BD$, where $B = \langle b \rangle$. Since $P$ acts transitively on $A$, it follows that every element of $H$ of order 3 is conjugate in $G$ to an element of $N$. Also, since $N$ contains a Sylow 2-subgroup of $H$, it follows that every element of $H$ whose order is a power of 2 is conjugate in $G$ to an element of $N$. We thus obtain that $\mathscr{P}_G(H) = \mathscr{P}_G(N)$.

Let $E/k$ be a Galois extension of algebraic number fields with the Galois group isomorphic to $G = AP$ defined above. Let $K$ and $L$ be the fixed fields of $H$ and $N$, respectively. The equality $\mathscr{P}_G(H) = \mathscr{P}_G(N)$ implies $N(K/k) = N(L/k)$ [12, Corollary 4, p. 344]. By Satz 1 of [1, p. 465], HNP holds for $K/k$. So $N_{K/k}K^* = N_{L/k}L^*$ iff HNP holds for $L/k$. We will use Theorem 3 of [3, p. 300] to show that $N(E/k)$ is contained in $N_{L/k}L^*$. For the convenience of the reader we will state this theorem here.

THEOREM 2.2 [3]. *Let $k \subseteq T \subseteq F$ be a tower of finite extensions of an algebraic number field $k$ with $F$ Galois over $k$. Let $G = G(F/k)$ and $H = G(F/T)$. If $G$ contains subgroups $G_1, ..., G_n$, and subgroups $H_s \subseteq G_s \cap H$ $(s = 1, ..., n)$ such that*

$$\prod_{s=1}^{n} \mathrm{Cor}_G^{G_s} \colon \prod_{s=1}^{n} \hat{H}^{-3}(G_s, \mathbf{Z}) \to \hat{H}^{-3}(G, \mathbf{Z})$$

*is a surjective homomorphism, and*

$$i(T_s/k_s) = 1 \qquad \textit{for each} \quad s = 1, ..., n,$$

*where $G(F/k_s) = G_s$ and $G(F/T_s) = H_s$ $(s = 1, ..., n)$, then $N(F/k) \subseteq N_{T/k}T^*$.*

Let $G_1$ be a Sylow 2-subgroup of $G$ containing $D$, and let $G_2 = A$. Since the $p$-primary components $(p = 2,3)$ of $\hat{H}^{-3}(G, \mathbf{Z})$ are homomorphic images of $\hat{H}^{-3}(G_s, \mathbf{Z})$ $(s = 1, 2)$ under the corestriction mappings [13, Proposition 3.1.15, p. 92], it follows that the homomorphism of cohomology groups in Theorem 2.2 is surjective. We set $N_s = G_s \cap N$ for each $s = 1, 2$. Let $k_s$ and $T_s$ be the fixed fields of $G_s$ and $N_s$, respectively. The Galois group of the normal closure of $T_1$ over $k_1$ is isomorphic to $\mathrm{D}_8$. By Satz 1 of [1, p. 465], HNP holds for $T_1/k_1$. Also, HNP holds for $T_2/k_2$, since this extension is cyclic. So by Theorem 2.2, $N(E/k) \subseteq N_{L/k}L^*$. It follows that the equality $N_{K/k}K^* = N_{L/k}L^*$ holds iff the factor group of $N(L/k)$ by $N(E/k)N_{L/k}L^*$ is trivial. By Theorem 1.11 this factor group is a homomorphic image of $N \cap G'/\Phi^G(N)$. Since the commutator subgroup $G' = A\langle x^2\rangle$, it follows that $N \cap G' = B\langle x^4\rangle$. To prove that $\Phi^G(N)$ contains elements $b$ and $x^4$ we note first that $x^{-4}bx^4 = b^2$ and $y = (x^2y)(yx^4)(x^2y)^{-1}$. So $b \in N \cap x^4Nx^{-4}$ and $y \in N \cap (x^2y)\,N(x^2y)^{-1}$. Since $b = [b, x^4]$ and $x^4 = [y, x^2y]$, it follows that $\Phi^G(N)$ contains $b$ and $x^4$. So $N \cap G' = \Phi^G(N)$, and therefore by Theorem 1.11 the factor group of $N(L/k)$ by $N(E/k)\,N_{L/k}L^*$ is trivial. This implies that the equality $N_{K/k}K^* = N_{L/k}L^*$ holds. Moreover, $L$, being an extension of $K$ of degree 3, is obviously not conjugate to $K$. Finally, since $G$ is a semidirect product of $A$ and $P$, it follows by Ikeda's theorem [8, p. 93] that any Galois extension $F/k$ with the Galois group $P$ can be embedded into a Galois extension $E/k$ with the Galois group $G$. We thus proved the following proposition.

PROPOSITION 2.3. *Let $F/k$ be a Galois extension of algebraic number fields with the Galois group isomorphic to a Sylow* 2*-subgroup of* GL(2, 3). *Let $k \subset K \subset F$ be a non-Galois extension of degree* 4 *such that $G(F/K)$ contains the center of $G(F/k)$. Then $K$ is not $k$-solitary.*

We note that $K/k$ in Proposition 2.3 is a non-Galois extension of the minimal degree that is not solitary over $k$. We wish now to consider non-Galois extensions of degree 5. Let $K/k$ be such an extension. If the Galois group of the normal closure of $K$ over $k$ is isomorphic to $S_5$, then $K$ is $k$-solitary. Indeed, the proof is similar to that in the case when the Galois group is $S_4$ (see the proof of Proposition 2.1).

An interesting case arises when the Galois group $G$ of the normal closure $E$ of $K$ over $k$ is isomorphic to $A_5$. Since any extension of $k$ of degree 5 contained in $E$ is conjugate to $K$ over $k$, we may assume without loss of generality that $K$ is the fixed field of $H \cong A_4$ generated by $\{(2,3)(4,5), (2,4)(3,5), (3,4,5)\}$. Let $k \subset L \subset E$ be the fixed field of $N \cong S_3$ generated by $\{(1,2)(4,5), (3,4,5)\}$. Since the cyclic group $\langle (3,4,5)\rangle$ acts transitively on the nonidentity elements of the Sylow 2-subgroup of $H$, which is also a Sylow 2-subgroup of $G$, it follows that $\mathscr{P}_G(H) = \mathscr{P}_G(N)$. So by Corollary 4 of [12, p. 344], $N(K/k) = N(L/k)$. For each prime $\upsilon$ of $k$ we denote by $G_\upsilon$ a decomposition group of $\upsilon$ in $E$. If $G_\upsilon$ is cyclic for each prime $\upsilon$ of $k$, then by Theorem 1.11

$$H \cap G'/\Phi^G(H) \cong N(K/k)/N(E/k)\,N_{K/k}K^*$$

and

$$N \cap G'/\Phi^G(N) \cong N(L/k)/N(E/k)\,N_{L/k}L^*.$$

Since $G = G'$, $\Phi^G(H) = H$, and $\Phi^G(N) = H \cap N = \langle (3, 4, 5) \rangle$, it follows that the factor group corresponding to $K/k$ is trivial, and the factor group corresponding to $L/k$ is of order 2. In particular, $N_{K/k} K^* \neq N_{L/k} L^*$. An example of a Galois extension such that $G_v$ is cyclic for each prime $v$ of the base field can be constructed using the polynomial [10, p. 285]

$$f(x) = x^5 + 2x^4 - 3x^3 - 5x^2 + x + 1.$$

The discriminant $D$ of $f(x)$ is a prime number 36,497. It follows by [10] that the Galois group of the splitting field $E$ over $\mathbf{Q}$ is isomorphic to $S_5$, and $E$ is an unramified extension of $k = \mathbf{Q}\ (\sqrt{D})$. So for the fixed fields $k \subset K$, $L \subset E$ of $H$ and $N$, respectively, we obtain $N_{K/k} K^* \neq N_{L/k} L^*$. We wish now to construct a Galois extension $E/k$ with the Galois group isomorphic to $A_5$ such that $N_{K/k} K^* = N_{L/k} L^*$. To construct such an extension it is necessary that at least one of the decomposition groups $G_v$ is not cyclic. Suppose that $E/k$ is a Galois extension with the Galois group $G$ isomorphic to $A_5$, and such that there is a prime $v$ of $k$ for which $G_v$ contains a Sylow 2-subgroup of $G$. It follows by Chebotarev's density theorem that for each prime $p$ dividing the order of $G$ a Sylow $p$-subgroup of $G$ is contained in a decomposition group. By Theorem 2.5 of [4, p. 18] HNP holds for $E/k$. Let $K$ and $L$ be the fixed fields of $H \cong A_4$ and $N \cong S_3$, respectively, as they were defined above. By Proposition 1 of [5, p. 314] HNP holds for $K/k$. In fact, HNP holds for any extension of prime degree [2, Lemma 4, p. 196; 9, Proposition 10.11, p. 44]. So to prove that $N_{K/k} K^* = N_{L/k} L^*$ it suffices to show that HNP holds for $L/k$. Moreover, since HNP holds for $E/k$, it follows that $N(E/k) = N_{E/k} E^* \subset N_{L/k} L^*$. We thus obtain that HNP holds for $L/k$ iff the factor group of $N(L/k)$ by $N(E/k) N_{L/k} L^*$ is trivial. By Theorem 1.11 this factor group is trivial iff $N = \Phi^G(N)\ X_{L/k}(E, E)$. We note that every subgroup of $A_5$ that contains a Sylow 2-subgroup is conjugate in $A_5$ to one of the following subgroups: $V_4 = \langle (2, 3)(4, 5), (2, 4)(3, 5) \rangle$ ($V_4$ is the Klein 4-group), $H$, or $A_5$. Moreover, since every decomposition group is a solvable group, it follows that either $G_v = V_4$, or $G_v = H$. If $G_v = \langle (2, 3)(4, 5), (2, 4)(3, 5) \rangle$, then $G$ is the union of 4 distinct double cosets $N \sigma_i G_v$, where $\sigma_1 = 1$, $\sigma_2 = (1, 2, 3)$, $\sigma_3 = (1, 2, 4, 5, 3)$, and $\sigma_4 = (1, 2, 5, 4, 3)$. So there are 4 distinct primes $\omega_i$ of $L$ above $v$. Let $N_{\omega_i} = N \cap \sigma_i G_v \sigma_i^{-1}$ be a decomposition group of $\omega_i$ in $E$. It follows that one of the decomposition groups is trivial and the remaining decomposition groups $N_{\omega_i} = \langle (1, 2)(4, 5) \rangle$ for $i = 2, 3, 4$. In the notation of the diagram (11) the kernel of

$$\psi_v \colon \prod_{i=1}^{4} N_{\omega_i} \to G_v$$

contains a 4-tuple in which the first component is 1, and the remaining components are equal to $(1,2)(4,5)$. So $\lambda_v$ (see the remark following Theorem 1.7) of this 4-tuple is equal to $(1,2)(4,5)N'$. So $X_{L/k}(E,E)$ contains $(1,2)(4,5)$, and therefore $N = \Phi^G(N)\, X_{L/k}(E,E)$. This shows that $N_{K/k}K^* = N_{L/k}L^*$. If $G_v = H$, then $G = NG_v \cup N(1,2,3)G_v$. So in this case there are two primes $\omega_i$ of $L$ above $v$ with decomposition groups $N_{\omega_1} = \langle (3,4,5) \rangle$ and $N_{\omega_2} = \langle (1,2)(4,5) \rangle$. The kernel of

$$\psi_v \colon N_{\omega_1} \times N_{\omega_2} \to G_v/G'_v$$

contains $N_{\omega_2}$. The image of $N_{\omega_2}$ under $\lambda_v$ contains $(1,2)(4,5)N'$. So $N = \Phi^G(N)\, X_{L/k}(E,E)$, and therefore the equality $N_{K/k}K^* = N_{L/k}L^*$ holds in the case when $G_v = H$. Finally we note that all extensions of $k$ of degree 10 contained in $E$ are conjugate to $L$ over $k$. We thus proved the following proposition.

Proposition 2.4. *Let $K/k$ be a finite extension of algebraic number fields of degree 5 whose normal closure $E$ over $k$ has the Galois group $G$ isomorphic to* $A_5$. *If there is a prime $v$* of *$k$ for which $G_v$ contains a Sylow 2-subgroup of $G$, then for any extension $k \subset L \subset E$ of degree 10 over $k$ the equality $N_{K/k}K^* = N_{L/k}L^*$ holds. In particular, $K$ is not $k$-solitary.*

In [10, p. 287] an irreducible polynomial $g(x) \in \mathbf{Z}[x]$ of degree 5 is constructed. It is shown in [10] that the Galois group $G$ of the splitting field $E$ of $g(x)$ over $\mathbf{Q}$ is isomorphic to $A_5$, and a decomposition group $G_p$ at the prime $p = 3$ coincides with a Sylow 2-subgroup of $G$. If $K, L \subset E$ are extensions of $k = \mathbf{Q}$ of degrees 5 and 10, respectively, then by Proposition 2.4 $N_{K/k}K^* = N_{L/k}L^*$.

We wish now to determine when two extensions $K/k$ and $L/k$ of algebraic number fields of degree 6 have equal norm groups. If one of these extensions, say $K/k$, is Galois, then $N_{K/k}K^* = N_{L/k}L^*$ implies $K \subseteq L$ [11, Corollary 1.9, p. 120). So in this case the equality of norm groups is equivalent to the equality of the extensions. We will assume, therefore, that both extensions are not Galois over $k$. Furthermore, we will assume that the Galois groups of the normal closures $\bar{K}$ and $\bar{L}$ of $K$ and $L$, respectively, over $k$ are isomorphic to $A_4$. We will use Theorem 2.2 to show that $N(\bar{K}/k) \subseteq N_{K/k}K^*$. Indeed, let $G = G(\bar{K}/k)$, $H = G(\bar{K}/K)$, and let $G_1$, $G_2$ be Sylow 2 and 3 subgroups of $G$, respectively. By Proposition 3.1.15 of [13, p. 92] the homomorphism of cohomology groups in Theorem 2.2 is surjective. If we denote by $F$ and $R$ the fixed fields of $G_1$ and $G_2$, respectively, then HNP holds for $K/F$ and $\bar{K}/R$ ($F$ is contained in $K$, since $G$ contains only one Sylow 2-subgroup). By Theorem 2.2 with $H_1 = H$ and $H_2 = 1$ we obtain that $N(\bar{K}/k) \subseteq N_{K/k}K^*$. It follows that for any finite Galois extension $E/k$ containing $K$, $N(E/k) \subseteq N_{K/k}K^*$. In [12] we constructed two extensions $K/k$ and $L/k$ of degree 6 with the Galois groups of their normal closures isomorphic to $A_4$ such that $N(K/k) = N(L/k)$, and the total obstructions to HNP for these extensions are both of order 2. Since for any finite Galois extension $E/k$ containing $K$ $(L)$ $N(E/k)$ is contained in $N_{K/k}K^*$ $(N_{L/k}L^*)$, the results in [3] on the first obstructions to HNP corresponding to towers of field extensions cannot be applied here to determine whether the equality $N_{K/k}K^* = N_{L/k}L^*$ holds. On the other hand the following proposition shows that the more general notion of the first obstructions to HNP corresponding to field extensions can be used to determine whether the above equality of norm groups holds.

PROPOSITION 2.5. *Let $K/k$ and $L/k$ be distinct extensions of algebraic number fields of degree* 6. *Suppose that the Galois groups of the normal closures $\bar{K}$ and $\bar{L}$ of $K$ and $L$, respectively, over $k$ are isomorphic to* $A_4$. *Then $N_{K/k}K^* = N_{L/k}L^*$ if and only if*

(a) $K \cap L \neq k$

(b) *there exist primes $\upsilon$ and $\omega$ of $k$ (not necessarily distinct) such that* 4 *divides the degrees of the extensions $\bar{K}_\upsilon/k_\upsilon$ and $\bar{L}_\omega/k_\omega$.*

*Proof.* We will first assume that the conditions (a) and (b) are satisfied. Since $A_4$ does not contain subgroups of order 6, it follows that the degree of $F = K \cap L$ over $k$ is equal to 3. By Proposition 1.1 of [11, p. 111], $N_{K/k}K^* = N_{F/k}F^* = N_{L/k}L^*$.

To prove that the equality of norm groups implies (a) and (b) we set $\bar{G} = G(\bar{K}/k)$ and $\bar{H} = G(\bar{K}/K)$. Since $\mathscr{P}_{\bar{G}}(V_4) = \mathscr{P}_{\bar{G}}(\bar{H})$ ($V_4$ is the Klein 4-group), it follows that $N_{K/k}J_K = N_{F_1/k}J_{F_1}$ [12, Theorem 3, p. 343], where $F_1$ is the fixed field of $V_4$. Similarly, $N_{L/k}J_L = N_{F_2/k}J_{F_2}$ with $k \subset F_2 \subset L$ being the fixed field of $V_4$. By Theorem 3 of [12, p. 343] $N_{K/k}K^* = N_{L/k}L^*$ implies $N_{K/k}J_K = N_{L/k}J_L$. So $N_{F_1/k}J_{F_1} = N_{F_2/k}J_{F_2}$. Since both extensions $F_i/k$ $(i = 1, 2)$ are Galois, the above equality of norm groups of idele groups implies $F_1 = F_2 = K \cap L$ [12]. In particular, $K \cap L \neq k$.

Suppose that 4 does not divide $(\bar{K}_\upsilon : k_\upsilon)$ for any prime $\upsilon$ of $k$. It follows that all local extensions of $\bar{K}/k$ are cyclic. So by Theorem 1.11 the first obstruction to HNP for $K/k$ corresponding to $\bar{K}/k$ is isomorphic to the factor group of $\bar{H} \cap \bar{G}'$ by $\Phi^{\bar{G}}(\bar{H}, 1) = \Phi^{\bar{G}}(\bar{H})$. Since $\bar{H} \subset \bar{G}'$ and $N_{\bar{K}/k}J_{\bar{K}} \subset N_{K/k}J_K$, it follows that

$$N(K/k)/N(\bar{K}/k)\, N_{K/k}K^* \cong \bar{H}/\Phi^{\bar{G}}(\bar{H}). \tag{22}$$

The factor group in the right side of (22) is of order 2. We already mentioned above that the equality $N_{K/k}J_K = N_{L/k}J_L$ holds. It follows that $N(K/k) = N(L/k)$. So by (22) $N(L/k)/N(\bar{K}/k)\, N_{L/k}L^*$ is also a group of order 2. Let $E$ be the compositum of $\bar{K}$ and $\bar{L}$. We set $G=G(E/k)$, $R=G(E/\bar{K})$, and $N=G(E/L)$. By Theorem 1.11 the first obstruction to HNP for $L/k$ corresponding to $\bar{K}/k$ is isomorphic to the factor group of $N \cap RG'$ by $\Phi^G(N, R)\, X_{L/k}(\bar{K}, E)$. Since $G'$ is the Sylow 2-subgroup of $G$, it follows that $N \cap RG' = N$. Also, $N_{\bar{K}/k} J_{\bar{K}} \subset N_{K/k} J_K = N_{L/k} J_L$ implies that the first obstruction to HNP for $L/k$ corresponding to $\bar{K}/k$ coincides with the factor group of $N(L/k)$ by $N(\bar{K}/k)\, N_{L/k} L^*$. Since this factor group is of order 2, it follows that the factor group of $N$ by $\Phi^G(N, R)\, X_{L/k}(\bar{K}, E)$ is of order 2. We wish now to compute the group $\Phi^G(N, R) = \Phi^G(N) \cdot \langle N \cap \mathscr{P}_G(R) \rangle$. Since $R$ is a normal 2-subgroup of $G$, it follows that $\mathscr{P}_G(R) = R$, and therefore $N \cap \mathscr{P}_G(R) = G(E/\bar{K}L)$. On the other hand $\Phi^G(N) = G(E/\bar{L})$. It follows that $\Phi^G(N, R) = N$, and therefore the factor group of $N$ by $\Phi^G(N, R)\, X_{L/k}(\bar{K}, E)$ is trivial, contradiction. We thus obtain that there is a prime $\upsilon$ of $k$ such that 4 divides $(\bar{K}_\upsilon : k_\upsilon)$. Similarly, there is a prime $\omega$ of $k$ such that 4 divides $(\bar{L}_\omega : k_\omega)$. ∎

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